STA 131 B: Mathematical Statistics	**Spring 2024**

Homework 1

Lecturer: Hang Zhou — *Due date: 4:00pm, April 11*

Homework is a crucial step in your learning journey for this course, enriching your understanding of mathematical statistics. I strongly suggest you spend time on it and complete it independently.

Question 1: Consider the multivariate normal distribution $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu}$ is a p-dimensional vector and Σ is a $p \times p$ covariance matrix. What is the parameter space for $\boldsymbol{\mu}$ and Σ?
Solution: The parameter space for μ is $\mathbb{R}$ and the parameter space for Σ is space that contains all $p \times p$ positive definite symmetric matrices.

Question 2: $X_1, \ldots, X_n \sim_{i.i.d.} \mathcal{N}(\mu, 1)$. Find the MLE of μ.
Solution: The probability density function for $X \sim \mathcal{N}(\mu, 1)$ is

$$f(x|\mu) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2}.$$

The likelihood function is

$$\tilde{L}(\mu) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^n (x_i-\mu)^2/2}.$$

The log likelihood function is

$$L(\mu) = -\frac{n}{2}\log 2\pi - \sum_{i=1}^n (x_i - \mu)^2/2.$$

Solve the zero point of first order derivative of $L(\mu)$,

$$\frac{dL(\mu)}{d\mu} = \sum_{i=1}^n (x_i - \mu) = 0 \Longrightarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^n X_i = \bar{X}.$$

Check the second order derivative of $L(\mu)$ at $\mu = \hat{\mu}$ is negative:

$$\left.\frac{d^2 L(\mu)}{d\mu^2}\right|_{\mu=\hat{\mu}} = -n < 0.$$

Thus, the MLE for μ is $\hat{\mu} = \bar{X}$.

Question 3: $X_1, \ldots, X_n \sim_{i.i.d.} \mathcal{N}(0, \sigma^2)$. Find the MLE of σ^2.
Solution: The probability density function for $X \sim \mathcal{N}(0, \sigma^2)$ is

$$f(x|\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}.$$

The likelihood function is

$$\tilde{L}(\sigma^2) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\sum_{i=1}^n x_i^2/(2\sigma^2)}.$$

The log likelihood function is

$$L(\sigma^2) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \sum_{i=1}^n \frac{x_i^2}{2\sigma^2}.$$

Solve the zero point of first order derivative of $L(\sigma^2)$,

$$\frac{dL(\sigma^2)}{d(\sigma^2)} = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n} x_i^2 = 0 \Longrightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

Check the second order derivative of $L(\mu)$ at $\mu = \hat{\mu}$ is negative:

$$\left.\frac{d^2L(\sigma^2)}{d(\sigma^2)^2}\right|_{\sigma^2=\hat{\sigma}^2} = \frac{1}{\hat{\sigma}^4}\left(\frac{n}{2} - \frac{1}{\hat{\sigma}^2}\sum_{i=1}^{n} X_i^2\right) = -\frac{n}{2}\frac{1}{\hat{\sigma}^4} < 0.$$

Thus, the MLE for σ^2 is $\hat{\sigma}^2 = n^{-1}\sum_{i=1}^{n} X_i^2$.

Question 4: A certain type of electronic component has a lifetime Y (in hours) with probability density function given by

$$f(y|\theta) = \begin{cases} \left(\frac{1}{\theta^2}\right) y e^{-y/\theta}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

That is, Y has a gamma distribution with parameters $\alpha = 2$ and θ. Find the MLE of θ.
Solution: The likelihood function is

$$\tilde{L}(\theta) = \left(\frac{1}{\theta^2}\right)^n e^{-\frac{\sum_{i=1}^{n} y_i}{\theta}} \prod_{i=1}^{n} y_i.$$

The log likelihood function is

$$L(\theta) = -2n\log\theta - \frac{1}{\theta}\sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\log y_i.$$

Solve the zero point of first order derivative of $L(\theta)$

$$\frac{dL(\theta)}{d\theta} = -\frac{2n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} y_i = 0 \Longrightarrow \hat{\theta} = \frac{1}{2n}\sum_{i=1}^{n} Y_i.$$

Check the second order derivative of $L(\theta)$ at $\theta = \hat{\theta}$,

$$\left.\frac{d^2L(\theta)}{d\theta^2}\right|_{\theta=\hat{\theta}} = \frac{2n}{\hat{\theta}^2} - \frac{2}{\hat{\theta}^3}\sum_{i=1}^{n} y_i = -\frac{2n}{\hat{\theta}^2} < 0$$

Thus, the MLE for θ is $\hat{\theta} = (2n)^{-1}\sum_{i=1}^{n} Y_i$.