STA 131 B: Mathematical Statistics	**Spring 2024**

Homework 5

Lecturer: Hang Zhou — *Due date: 2:00pm, May 23*

Homework is a crucial step in your learning journey for this course, enriching your understanding of mathematical statistics. I strongly suggest you spend time on it and complete it independently.

Question 1: Suppose we have random samples $X_1, \ldots, X_n$ from a Uniform($[0, \theta]$) distribution where θ is the unknown parameter. Recall $X_{(n)} = \max_i X_i$ is the order statistic. [Your total grade for this question is capped at 40. However, please do your best to earn as many points as possible!]

a). [10 points] Show that the CDF of $X_{(n)}$ is $\left(\frac{x}{\theta}\right)^n$.
[*hint*]: Imagine a random sample falling in such a way that the maximum is below a fixed value x. This will happen if and only if all of the X_i are below x, i.e., $F_{X_{(n)}}(x) = P\left(X_{(n)} \leq x\right) = P\left(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x\right)$, and X_i are independent.

b). [10 points] Use a). to show that the pdf of $X_{(n)}$ is nx^{x-1}/θ^n.

c). [15 points] Consider the MoM estimator of θ, that is, $\hat{\theta}_1 = 2\bar{X}$, where $\bar{X} = n^{-1}\sum_{i=1}^n X_i$. Calculate $\mathbb{E}(\hat{\theta}_1)$, $\text{var}(\hat{\theta}_1)$ and $\text{MSE}_\theta(\hat{\theta}_1)$.
[*hint*]: If $X \sim \text{Uniform}([a, b])$, $\mathbb{E}(X) = (a + b)/2$, $\text{var}(X) = (b - a)^2/12$.

d). [15 points] The maximum likelihood estimator is $\hat{\theta}_2 = X_{(n)}$. Calculate $\mathbb{E}(\hat{\theta}_2)$, $\text{var}(\hat{\theta}_2)$ and $\text{MSE}_\theta(\hat{\theta}_2)$.
[*hint*]: Use the result in b) and $\text{var}(X_{(n)}) = \mathbb{E}(X_{(n)}^2) - (\mathbb{E}X_{(n)})^2$.

e). [Bonus Question, 15 points]: Find the constant c such that $cX_{(n)}$ has the minimum MSE over the class $\{cX_{(n)} : c \in \mathbb{R}\}$.
[*hint*]: $x^* = -b/2a$ minimize the $ax^2 + bx + c$ over all x.